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Atomic coherent states studied by virtue of the EPR entangled state and their Wigner functions*

Hong-yi Fan[1,2,3,a] and J. Chen[3]

[1] CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, P.R. China
[2] Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, P.R. China
[3] Department of Material Science and Engineering, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China

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Abstract. Based on the newly constructed Einstein, Podolsky and Rosen (EPR) entangled state representation we introduce macroscopic classical functions associated with atomic coherent state $|\tau\rangle$ with angular momentum value j. These functions are proportional to the ordinary one-variable Hermite polynomials of order $2j$. The corresponding Wigner quasiprobability function for $|\tau\rangle$ in phase space is also derived which turns out to be a two-variable Hermite polynomial $H_{2j,2j}$. In so doing, a new classical-quantum correspondence scheme for angular momentum system is established.

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In recent years quantum entanglement and entangled states have been paid much attention by physicists due to their weird and fascinating properties. The idea of quantum entanglement stemmed from Einstein-Podolsky-Rosen (EPR) who argued that quantum mechanics is incomplete [1]. The corresponding entangled states in two-mode Fock space were constructed in references [2,3] which make up orthonormal and complete representations. In this work we aim at employing the EPR entangled state representation to derive the Wigner function of atomic coherent states (which are sometimes referred to in the literature as spin coherent state or Bloch state) [4–6]. The atomic coherent states have successfully applied in many branches of physics [7–9]; for example, Narducci, Bowden, Bluemel, Garrazana and Tuft [7] used atomic coherent state to study multitime correlation function for systems with observables satisfying an angular momentum algebra, which suggested a convenient classical-quantum correspondence rule for angular momentum degrees of freedom. Arecchi *et al.* applied atomic coherent states to describe interactions between radiation field and an assembly of two-level atoms [4]. Takahashi and Shibata [9] transformed some equation of motion for density matrix of a damped spin system into that of a quasi-distribution. However, to our knowledge, how to successfully introduce the Wigner function of the spin coherent states has not been reported in the literature before. In quantum optics theory the Wigner function of a state is defined as the expectation value of the Wigner operator in this state [10]. As is well-known, the Wigner quasidistribution provides with a definite phase space distribution of quantum states and is very useful in quantum statistics and quantum optics [11–13]. Thus we are motivated to calculate the Wigner function of the atomic coherent state $|\tau\rangle$ in a similar way as in quantum optics. We shall employ the Schwinger Bose realization of spin operators [14] to construct the atomic coherent states for different angular momentum values. Du to the entanglement involved in $|\tau\rangle$ (see Eq. (17) below, which can be viewed as a Schmidt decomposition of an entangled state), we shall introduce the EPR entangled state to study the state $|\tau\rangle$. Our work is arranged as follows: in Section 2 we briefly recall the properties of (EPR) entangled state $\langle\xi|$. In Section 3 we calculate the wave function of atomic coherent states $|\tau\rangle$ with definite angular momentum quantum number j in $\langle\xi|$ representation, the result is a new function proportional to an ordinary Hermite polynomial of order $2j$ associated with $|\tau\rangle$. In Section 4 we employ the Wigner operator in $\langle\xi|$ representation to derive the Wigner function of $|\tau\rangle$. It turns out that the Wigner function of $|\tau\rangle$ is a two-variable Hermite polynomial of order $(2j, 2j)$. This concise expression bring us much convenience to study the relation between different atomic coherent states with different j, since the recursive relation of the Hermite polynomials are already known. In Section 5 we use the result in Section 3 to show that some Hamiltonian of angular momentum system possesses the atomic coherent state $|\tau\rangle$ as its

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[a] e-mail: `fhym@sjtu.edu.en`

eigenstate, and the value of τ and the energy eigenvalue can be determined.

1 EPR entangled state

We introduce the EPR eigenstate [2,3]

$$\langle\zeta| = \langle 00| \exp\{-ab + a\zeta + b\zeta^*\}, \tag{1}$$

which obey the eigenvector equations

$$\langle\zeta| \left(a^\dagger + b\right) = \zeta \langle\zeta|, \quad \langle\zeta| \left(a + b^\dagger\right) = \zeta^* \langle\zeta|, \tag{2}$$

and

$$\frac{\partial}{\partial\zeta} \langle\zeta| = \langle\zeta| a, \quad \frac{\partial}{\partial\zeta^*} \langle\zeta| = \langle\zeta| b. \tag{3}$$

We can see that in the $\langle\zeta|$ basis,

$$\begin{aligned} a &\to \frac{\partial}{\partial\zeta}, \quad a^\dagger \to \zeta - \frac{\partial}{\partial\zeta^*}, \\ b &\to \frac{\partial}{\partial\zeta^*}, \quad b^\dagger \to \zeta^* - \frac{\partial}{\partial\zeta}. \end{aligned} \tag{4}$$

Using the relation between coordinate (momentum) operator and the Bose creation-annihilation operators

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{2}} \left(a + a^\dagger\right), \quad X_2 = \frac{1}{\sqrt{2}} \left(b + b^\dagger\right), \\ P_1 &= \frac{1}{\mathrm{i}\sqrt{2}} \left(a - a^\dagger\right), \quad P_2 = \frac{1}{\mathrm{i}\sqrt{2}} \left(b - b^\dagger\right), \end{aligned} \tag{5}$$

we see that $|\zeta\rangle$ is the common eigenvector of $X_1 + X_2$ and $P_1 - P_2$,

$$\begin{aligned} (X_1 + X_2) |\zeta\rangle &= \sqrt{2}\zeta_1 |\zeta\rangle, \\ (P_1 - P_2) |\zeta\rangle &= \sqrt{2}\zeta_2 |\zeta\rangle. \end{aligned} \tag{6}$$

Recall the concept of EPR entanglement we can say that $|\zeta\rangle$ is an entangled state of continuous variables. Using the normal product form of two-mode vacuum state

$$|00\rangle \langle 00| =: \exp\{-a^\dagger a - b^\dagger b\} :, \tag{7}$$

and the technique of integration within an ordered product of operators [15,16] of operators we can prove the completeness relation

$$\int \frac{\mathrm{d}^2\zeta}{\pi} \mathrm{e}^{-|\zeta|^2} |\zeta\rangle \langle\zeta| = \mathbf{1} \tag{8}$$

and the orthonormal relation

$$\langle\zeta \,|\zeta'\rangle = \pi\delta\left(\zeta - \zeta'\right) \delta\left(\zeta^* - \zeta'^*\right). \tag{9}$$

2 Wave function of Atomic coherent state in $|\zeta\rangle$ representation

The atomic coherent state is defined as [4–7]

$$\begin{aligned} |\tau\rangle &= \exp\{\mu J_+ - \mu^* J_-\} |j, -j\rangle \\ &= \left(1 + |\tau|^2\right)^{-j} \mathrm{e}^{\tau J_+} |j, -j\rangle, \end{aligned} \tag{10}$$

where J_+ is the raising operator of $|j, m\rangle$, $|j, -j\rangle$ is the lowest weight state annihilated by J_-, and

$$\mu = \frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi}, \quad \tau = \mathrm{e}^{-\mathrm{i}\varphi} \tan\frac{\theta}{2}, \tag{11}$$

which satisfy

$$\begin{aligned} \int \frac{\mathrm{d}\Omega}{4\pi} |\tau\rangle \langle\tau| &= \sum_m |j, m\rangle \langle j, m| = \mathbf{1}_j, \\ \mathrm{d}\Omega &= \sin\theta \mathrm{d}\theta\mathrm{d}\varphi \end{aligned} \tag{12}$$

and

$$\langle\tau'|\, \tau\rangle = \left(1 + \tau'\tau^*\right)^{2j} / \left(1 + |\tau|^2\right)^j \left(1 + |\tau'|^2\right)^j. \tag{13}$$

Using $[J_+, J_-] = 2J_z,\ [J_\pm, J_z] = \mp J_\pm$, one can show that the state $|\tau\rangle$ obey the following eigenvector equations

$$\begin{aligned} \left(J_- + \tau^2 J_+\right) |\tau\rangle &= 2j\tau |\tau\rangle, \\ \left(J_- + \tau J_z\right) |\tau\rangle &= j\tau |\tau\rangle, \\ \left(\tau J_+ - J_z\right) |\tau\rangle &= j |\tau\rangle. \end{aligned} \tag{14}$$

In this section we employ the $|\zeta\rangle$ representation to study the properties of $|\tau\rangle$, so that a new wave functions associated with the atomic coherent states can be introduced. For this purpose, we employ the Schwinger Bose operator realization of angular momentum

$$J_+ = a^\dagger b,\ J_- =\ b^\dagger a,\ \ J_z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right) \tag{15}$$

and

$$\begin{aligned} |j, m\rangle &= \frac{a^{\dagger j+m} b^{\dagger j-m}}{\sqrt{(j+m)!\,(j-m)!}} |00\rangle \\ &= |j+m\rangle \otimes |j-m\rangle, \end{aligned} \tag{16}$$

where the last ket is written in two-mode Fock space, then $|\tau\rangle$ is expressed as

$$\begin{aligned} |\tau\rangle &= \exp\{\mu J_+ - \mu^* J_-\} |0\rangle \otimes |2j\rangle \\ &= \frac{1}{\sqrt{(2j)!}} \left(b^\dagger \cos\frac{\theta}{2} + a^\dagger \mathrm{e}^{-\mathrm{i}\varphi} \sin\frac{\theta}{2}\right)^{2j} |00\rangle \\ &= \frac{1}{\left(1 + |\tau|^2\right)^j} \sum_{l=0}^{2j} \left[\frac{(2j)!}{l!\,(2j-l)!}\right]^{1/2} \tau^{2j-l} |2j-l\rangle \otimes |l\rangle, \end{aligned} \tag{17}$$

which is a Schmidt decomposition. Using (4) we see

$$
\begin{aligned}
\langle\zeta| J_+ &= \langle\zeta| a^\dagger b = \left(\zeta - \frac{\partial}{\partial\zeta^*}\right)\frac{\partial}{\partial\zeta^*}\langle\zeta|,\\
\langle\zeta| J_- &= \langle\zeta| b^\dagger a = \left(\zeta^* - \frac{\partial}{\partial\zeta}\right)\frac{\partial}{\partial\zeta}\langle\zeta|,\\
\langle\zeta| J_z &= \frac{1}{2}\langle\zeta|\left(a^\dagger a - b^\dagger b\right) = \frac{1}{2}\left(\zeta\frac{\partial}{\partial\zeta} - \zeta^*\frac{\partial}{\partial\zeta^*}\right)\langle\zeta|.
\end{aligned} \tag{18}
$$

The completeness relation (12) in j-subspace is now extend to the whole two-mode Fock space

$$
\sum_{2j=0}^{\infty}(2j+1)\int\frac{\mathrm{d}\Omega}{4\pi}|\tau\rangle\langle\tau| = 1, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, ... \tag{19}
$$

Using the definition of two-variable Hermite polynomial $H_{m,n}(\zeta,\zeta^*)$ [17,18]

$$
H_{m,n}(\zeta,\zeta^*) = \sum_{l=0}^{\min(m,n)}\frac{m!n!}{l!(m-l)!(n-l)!}(-1)^l\zeta^{m-l}\zeta^{*n-l}, \tag{20}
$$

(which is not a direct product of two independent single-variable Hermite polynomials) and the generating function of $H_{m,n}(\zeta,\zeta^*)$,

$$
\sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}H_{m,n}(\zeta,\zeta^*) = \exp\{-tt' + t\zeta + t'\zeta^*\}, \tag{21}
$$

we have

$$
\langle\zeta| = \sum_{m,n=0}^{\infty}\frac{a^m b^n}{m!n!}H_{m,n}(\zeta,\zeta^*), \tag{22}
$$

thus

$$
\langle\zeta| m,n\rangle = \frac{1}{\sqrt{m!n!}}H_{m,n}(\zeta,\zeta^*). \tag{23}
$$

It then follows from (16) and (23) that

$$
\langle\zeta|\tau\rangle = \frac{\sqrt{(2j)!}}{(1+|\tau|^2)^j}\sum_{l=0}^{2j}\frac{\tau^{2j-l}H_{2j-l,l}(\zeta,\zeta^*)}{l!\,(2j-l)!}. \tag{24}
$$

Using the integration expression of $H_{m,n}$,

$$
(-1)^n\mathrm{e}^{\xi\eta}\int\frac{\mathrm{d}^2z}{\pi}z^n z^{*m}\exp\{-|z|^2 + \xi z - \eta z^*\} = H_{m,n}(\xi,\eta), \tag{25}
$$

we can reform the sum in (24) as

$$
\begin{aligned}
\langle\zeta|\tau\rangle &= \frac{\sqrt{(2j)!}}{(1+|\tau|^2)^j}\sum_{l=0}^{2j}(-1)^l\mathrm{e}^{\zeta\zeta^*}\int\frac{\mathrm{d}^2z}{\pi}\frac{(\tau z^*)^{2j-l}z^l}{l!\,(2j-l)!}\\
&\quad\times\exp\{-|z|^2 + \zeta z - \zeta^* z^*\}\\
&= \frac{1}{(1+|\tau|^2)^j\sqrt{(2j)!}}\mathrm{e}^{\zeta\zeta^*}\int\frac{\mathrm{d}^2z}{\pi}(\tau z^* - z)^{2j}\\
&\quad\times\exp\{-|z|^2 + \zeta z - \zeta^* z^*\}.
\end{aligned} \tag{26}
$$

By making the integration variable transform

$$
\begin{aligned}
z - \tau z^* &= \left(1-|\tau|^2\right)z',\\
z = z' + \tau z'^*,&\ z^* = z'^* + \tau^* z',
\end{aligned} \tag{27}
$$

with the Jacobian being

$$
\mathrm{d}^2z = \frac{\mathrm{i}}{2}\mathrm{d}z\wedge\mathrm{d}z^* = \left(1-|\tau|^2\right)\mathrm{d}^2z', \tag{28}
$$

and letting

$$
s = \zeta - \tau^*\zeta^* \tag{29}
$$

equation (26) becomes

$$
\begin{aligned}
\langle\zeta|\tau\rangle &= \frac{1}{(1+|\tau|^2)^j\sqrt{(2j)!}}\mathrm{e}^{\zeta\zeta^*}\left(1-|\tau|^2\right)^{2j+1}\\
&\quad\times\left(-\frac{\partial}{\partial s}\right)^{2j}\int\frac{\mathrm{d}^2z'}{\pi}\exp\{-\left(1+|\tau|^2\right)|z'|^2\\
&\quad - \tau^* z'^2 - \tau z'^{*2} + sz' - s^* z'^*\}.
\end{aligned} \tag{30}
$$

With the help of mathematical formula

$$
\int\frac{\mathrm{d}^2z}{\pi}\exp\{\lambda|z|^2 + fz^2 + gz^{*2} + \xi z + \eta z^*\} = \frac{1}{\sqrt{\lambda^2 - 4fg}}\exp\left[\frac{-\lambda\xi\eta + \xi^2 g + \eta^2 f}{\lambda^2 - 4fg}\right], \tag{31}
$$

we have

$$
\begin{aligned}
\langle\zeta|\tau\rangle &= \frac{\mathrm{e}^{\zeta\zeta^*}\left(1-|\tau|^2\right)^{2j}}{(1+|\tau|^2)^j\sqrt{(2j)!}}\left(-\frac{\partial}{\partial s}\right)^{2j}\\
&\quad\times\exp\left[\frac{-\left(1+|\tau|^2\right)ss^* - \tau^* s^{*2} - \tau s^2}{\left(1-|\tau|^2\right)^2}\right].
\end{aligned} \tag{32}
$$

Using the expression of single-variable Hermite polynomial

$$
H_n(x) = \mathrm{e}^{x^2}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\mathrm{e}^{-x^2}, \tag{33}
$$

and setting

$$
\frac{\sqrt{\tau}}{1-|\tau|^2}s + \frac{1+|\tau|^2}{1-|\tau|^2}\frac{s^*}{2\sqrt{\tau}} \equiv \chi, \tag{34}
$$

equation (32) becomes

$$
\begin{aligned}
\langle\zeta|\tau\rangle &= \frac{\tau^j H_{2j}(\chi)}{(1+|\tau|^2)^j\sqrt{(2j)!}}\exp\left[\zeta\zeta^* - \frac{\tau^* s^{*2}}{\left(1-|\tau|^2\right)^2}\right.\\
&\quad\left. + \left(\frac{1+|\tau|^2}{1-|\tau|^2}\frac{s^*}{2\sqrt{\tau}}\right)^2 - \chi^2\right].
\end{aligned} \tag{35}
$$

Substituting (29) into (34) we notice $(\zeta^*+\tau\zeta)/(2\sqrt{\tau})=\chi$, then using (29) again we can see that the terms in the exponential of (35) vanishes, and

$$\langle\zeta\,|\tau\rangle=\frac{\tau^j}{(1+|\tau|^2)^j\sqrt{(2j)!}}H_{2j}\left(\frac{\zeta^*+\tau\zeta}{2\sqrt{\tau}}\right), \tag{36}$$

where H_{2j} is a single-variable ordinary Hermite polynomial, but with the complex argument χ. Equation (36) is a easily remembered macroscopic classical functions associated with $|\tau\rangle$. In order to check the correctness of (36), we introduce the un-normalized atomic coherent state

$$|\tau\rangle\rangle=\mathrm{e}^{\tau J_+}|j,-j\rangle, \tag{37}$$

with

$$\langle\zeta\,|\tau\rangle\rangle=\frac{\tau^j}{\sqrt{(2j)!}}H_{2j}(\chi),\quad \chi=\frac{\zeta^*+\tau\zeta}{2\sqrt{\tau}}\cdot \tag{38}$$

We examine if (38) satisfies the following equation,

$$\langle\zeta|\,J_+\,|\tau\rangle\rangle=\frac{\partial}{\partial\tau}\langle\zeta\,|\tau\rangle\rangle=\left(\zeta-\frac{\partial}{\partial\zeta^*}\right)\frac{\partial}{\partial\zeta^*}\langle\zeta\,|\tau\rangle\rangle. \tag{39}$$

Note

$$\frac{\partial}{\partial\tau}H_{2j}(\chi)=\frac{1}{4\tau^{3/2}}[\tau\zeta-\zeta^*]\frac{\partial}{\partial\chi}H_{2j}(\chi), \tag{40}$$

and the property of H_n

$$2\chi H_n'(\chi)-H_n''(\chi)=2nH_n(\chi), \tag{41}$$

and $(\zeta^*+\tau\zeta)/(2\sqrt{\tau})=\chi$ we do have

$$\begin{aligned}\left[\frac{\partial}{\partial\tau}-\left(\zeta-\frac{\partial}{\partial\zeta^*}\right)\frac{\partial}{\partial\zeta^*}\right]\left[\tau^jH_{2j}(\chi)\right]=&\\ j\tau^{j-1}H_{2j}(\chi)-\tau^{j-1}H_{2j}'(\chi)\frac{\zeta^*+\tau\zeta}{4\sqrt{\tau}}&\\ +\frac{1}{4}\tau^{j-1}H_{2j}''(\chi)=0.&\end{aligned} \tag{42}$$

Similarly, using (40, 41) we can prove

$$\left[\tau\frac{\partial}{\partial\tau}-\frac{1}{2}\left(\zeta\frac{\partial}{\partial\zeta}-\zeta^*\frac{\partial}{\partial\zeta^*}\right)\right]\left[\tau^jH_{2j}(\chi)\right]=j\tau^jH_{2j}(\chi), \tag{43}$$

and

$$\begin{aligned}&\left[\left(\zeta^*-\frac{\partial}{\partial\zeta}\right)\frac{\partial}{\partial\zeta}+\tau^2\frac{\partial}{\partial\tau}\right]\left[\tau^jH_{2j}(\chi)\right]=2j\tau^{j+1}H_{2j}(\chi),\\&\left[\left(\zeta^*-\frac{\partial}{\partial\zeta}\right)\frac{\partial}{\partial\zeta}+\frac{\tau}{2}\left(\zeta\frac{\partial}{\partial\zeta}-\zeta^*\frac{\partial}{\partial\zeta^*}\right)\right]\\&\qquad\times\left[\tau^jH_{2j}(\chi)\right]=j\tau^{j+1}H_{2j}(\chi),\end{aligned} \tag{44}$$

which correspond to (14) when (18) is used.

3 The Wigner function of atomic coherent state

We now derive the Wigner function for the atomic coherent states $|\tau\rangle$. Recall that the Wigner operator in the single-mode coherent state representation is [19]

$$\begin{aligned}\Delta(x_1,p_1)\to\Delta(\alpha)=\int\frac{\mathrm{d}^2z}{\pi}|\alpha+z\rangle\langle\alpha-z|&\\ \times\exp\{\alpha z^*-z\alpha^*\},&\\ \alpha=\frac{1}{\sqrt{2}}(x_1+\mathrm{i}p_1)&\end{aligned} \tag{45}$$

where $|z\rangle$ is the coherent state [20]

$$|z\rangle=\exp[-\frac{1}{2}|z|^2+za^\dagger]\,|0\rangle. \tag{46}$$

Since the atomic coherent states $|\tau\rangle$ is now defined in the two-mode Fock space, we should use two-mode Wigner operator to calculate the its Wigner function. It can be proved that the two-mode Wigner operator $\Delta(\alpha)\Delta(\beta)$ in the entangled state $\langle\zeta\|$ representation is given by [21] (see Appendix)

$$\begin{aligned}\Delta(\alpha)\Delta(\beta)=\int\frac{d^2\zeta}{\pi^3}\|\gamma-\zeta\rangle\langle\gamma+\zeta\|&\\ \times\exp\{\rho\zeta^*-\zeta\rho^*\}=\Delta(\gamma,\rho),&\\ \gamma=\alpha+\beta^*,\quad\rho=\alpha-\beta^*,\quad\beta=\frac{1}{\sqrt{2}}(x_2+\mathrm{i}p_2),&\end{aligned} \tag{47}$$

where $\|\zeta\rangle$ is related to $|\zeta\rangle$ by a normalization factor

$$\|\zeta\rangle=\mathrm{e}^{-\frac{|\zeta|^2}{2}}|\zeta\rangle. \tag{48}$$

Using (36) we see

$$\begin{aligned}\langle\tau|\,\Delta(\gamma,\rho)\,|\tau\rangle&=\langle\tau|\int\frac{\mathrm{d}^2\zeta}{\pi^3}\|\gamma-\zeta\rangle\langle\gamma+\zeta\|\exp\{\rho\zeta^*-\zeta\rho^*\}|\tau\rangle\\&=\frac{|\tau|^{2j}}{(2j)!\,(1+|\tau|^2)^{2j}}\int\frac{\mathrm{d}^2\zeta}{\pi^3}H_{2j}^*\left(\frac{(\gamma-\zeta)^*+\tau(\gamma-\zeta)}{2\sqrt{\tau}}\right)\\&\quad\times H_{2j}\left(\frac{(\gamma+\zeta)^*+\tau(\gamma+\zeta)}{2\sqrt{\tau}}\right)\\&\quad\times\exp\left\{-\frac{|\gamma-\zeta|^2+|\gamma+\zeta|^2}{2}+\rho\zeta^*-\zeta\rho^*\right\}.\end{aligned} \tag{49}$$

Making a integration variable transform,

$$\zeta\to\zeta'\equiv\frac{\zeta^*+\tau\zeta}{\sqrt{\tau}}, \tag{50}$$

with its inverse transform

$$\zeta=\sqrt{\tau^*}\frac{\zeta'^*-|\tau|\,\zeta'}{1-|\tau|^2}, \tag{51}$$

and setting $\gamma' \equiv \frac{\gamma^* + \tau\gamma}{\sqrt{\tau}}$ we can simplify (49) as

$$\frac{(2j)!\left(1+|\tau|^2\right)^{2j}}{|\tau|^{2j}} \mathrm{e}^{|\gamma|^2} \langle\tau| \Delta(\gamma,\rho) |\tau\rangle = \frac{|\tau|}{1-|\tau|^2} \int \frac{\mathrm{d}^2\zeta'}{\pi^3} H_{2j}\left(\frac{\gamma'^* - \zeta'^*}{2}\right) H_{2j}\left(\frac{\gamma' + \zeta'}{2}\right) \times \exp\left\{ -\left|\sqrt{\tau^*}\frac{\zeta'^* - |\tau|\zeta'}{1-|\tau|^2}\right|^2 + \rho\left(\sqrt{\tau}\frac{\zeta' - |\tau|\zeta'^*}{1-|\tau|^2}\right) - \left(\sqrt{\tau^*}\frac{\zeta'^* - |\tau|\zeta'}{1-|\tau|^2}\right)\rho^*\right\}. \quad (52)$$

Then using

$$H_n\left(\frac{x}{2}\right) = \left(\frac{\partial}{\partial t}\right)^n \exp\{tx - t^2\}|_{t=0}, \quad (53)$$

we see that (52) becomes to

$$\frac{(2j)!\left(1+|\tau|^2\right)^{2j}}{|\tau|^{2j}} \mathrm{e}^{|\gamma|^2} \langle\tau| \Delta(\gamma,\rho) |\tau\rangle = \frac{|\tau|}{1-|\tau|^2}\left(\frac{\partial}{\partial t}\right)^{2j} \times \left(\frac{\partial}{\partial t'}\right)^{2j} \int \frac{\mathrm{d}^2\zeta'}{\pi^3} \exp\{t(\gamma'^* - \zeta'^*) + t'(\gamma' + \zeta') - t^2 - t'^2 - \left|\sqrt{\tau^*}\frac{\zeta'^* - |\tau|\zeta'}{1-|\tau|^2}\right|^2 + \rho\left(\sqrt{\tau}\frac{\zeta' - |\tau|\zeta'^*}{1-|\tau|^2}\right) - \left(\sqrt{\tau^*}\frac{\zeta'^* - |\tau|\zeta'}{1-|\tau|^2}\right)\rho^*\}|_{t=t'=0}. \quad (54)$$

Using (31) to perform the integration over $\mathrm{d}^2\zeta'$ leads to the result

$$\int \frac{\mathrm{d}^2\zeta'}{\pi^3} \exp\left\{ \frac{-\left(1+|\tau|^2\right)|\zeta'|^2 + |\tau|\left(\zeta'^{*2} + \zeta'^2\right)}{\left(1-|\tau|^2\right)^2}|\tau| + \zeta'\left(t' + \frac{\sqrt{\tau}\rho}{1-|\tau|^2} + \frac{|\tau|\sqrt{\tau^*}\rho^*}{1-|\tau|^2}\right) + \zeta'^*\left(-t - \frac{|\tau|\sqrt{\tau}\rho}{1-|\tau|^2} - \frac{\sqrt{\tau^*}\rho^*}{1-|\tau|^2}\right)\right\} = \frac{1-|\tau|^2}{\pi^2|\tau|} \exp\left\{ t^2 + t'^2 - \frac{1+|\tau|^2}{|\tau|}tt' + \frac{1}{|\tau|}(wt + w^*t') - |\rho|^2\right\}, \quad (55)$$

where

$$w = 2|\tau| g - g^*\left(1+|\tau|^2\right) = \sqrt{\tau}\left(\tau^*\rho^* - \rho\right),\qquad g = \frac{|\tau|}{1-|\tau|^2}\frac{\rho^* + \tau\rho}{\sqrt{\tau}}. \quad (56)$$

Substituting (55) into (54) and using

$$H_{m,n}(\zeta,\zeta^*) = \frac{\partial^{m+n}}{\partial t^m \partial t'^n} \exp\{-tt' + t\zeta + t'\zeta^*\}|_{t=t'=0}, \quad (57)$$

we see that the Wigner function of $|\tau\rangle$ of angular momentum j is characteristic of a two-variable Hermite polynomial of order $(2j, 2j)$,

$$\begin{aligned}\langle\tau| \Delta(\gamma,\rho) |\tau\rangle &= \frac{|\tau|^{2j}}{\pi^2 (2j)!\left(1+|\tau|^2\right)^{2j}} \mathrm{e}^{-|\gamma|^2 - |\rho|^2} \times \left(\frac{\partial}{\partial t}\right)^{2j}\left(\frac{\partial}{\partial t'}\right)^{2j} \exp\{-\frac{1+|\tau|^2}{|\tau|}tt' + \frac{1}{|\tau|}(wt + w^*t') + t\gamma'^* + t'\gamma'\}|_{t=t'=0} \\ &= \frac{1}{\pi^2 (2j)!}\mathrm{e}^{-|\gamma|^2 - |\rho|^2} \times H_{2j,2j}\left[\lambda\left(\rho'^* + \gamma'^*\right), \lambda\left(\rho' + \gamma'\right)\right] \\ &= \frac{1}{\pi^2 (2j)!}\mathrm{e}^{-2\left(|\alpha|^2 + |\beta|^2\right)} \times H_{2j,2j}\left[2\lambda\frac{\tau^*\alpha^* + \beta^*}{\sqrt{\tau *}}, 2\lambda\frac{\tau\alpha + \beta}{\sqrt{\tau}}\right],\end{aligned} \quad (58)$$

where

$$\rho' = \frac{\tau\rho - \rho^*}{\sqrt{\tau}},\quad \lambda^2 = \frac{|\tau|}{1+|\tau|^2},\quad \gamma' \equiv \frac{\gamma^* + \tau\gamma}{\sqrt{\tau}}. \quad (59)$$

4 Atomic coherent state as an eigenstate of some Hamiltonian of angular momentum system

As an application of the above theory, we consider the Hermitian Hamiltonian

$$H = DJ_- + D^*J_+ + CJ_z. \quad (60)$$

Supposing the atomic coherent state $|\tau\rangle\rangle$ is an eigenstate of H with energy eigenvalue E,

$$\langle\zeta| H |\tau\rangle\rangle = E\langle\zeta |\tau\rangle\rangle \quad (61)$$

we want to determine the energy E and the value τ. According to equations (14, 18) we have

$$\begin{aligned}\langle\zeta|H|\tau\rangle\rangle &= \langle\zeta|\left[D\left(2j\tau - \tau^2 J_+\right) + D^*J_+ + C\left(\tau J_+ - j\right)\right]|\tau\rangle\rangle \\ &= \left(2j\tau D - jC\right)\langle\zeta|\tau\rangle\rangle + \left(D^* - \tau^2 D + C\tau\right)\langle\zeta| J_+ |\tau\rangle\rangle \\ &= \left[\left(2j\tau D - jC\right) + \left(D^* - \tau^2 D + C\tau\right) \times \left(\zeta^* - \frac{\partial}{\partial\zeta}\right)\frac{\partial}{\partial\zeta}\right]\langle\zeta |\tau\rangle\rangle.\end{aligned} \quad (62)$$

From equation (38) we see $\partial/\partial\zeta = (\sqrt{\tau}/2)\mathrm{d}/\mathrm{d}\chi$, and

$$\langle\zeta|H|\tau\rangle\rangle = \left[2j\tau D - jC + \left(D^* - \tau^2 D + C\tau\right) \times \left(\zeta^* - \frac{\sqrt{\tau}d}{2d\chi}\right)\frac{\sqrt{\tau}d}{2d\chi}\right]\langle\zeta|\tau\rangle\rangle = E\langle\zeta|\tau\rangle\rangle. \quad (63)$$

Since $\mathrm{d}H_{2j}(\chi)/\mathrm{d}\chi = 4jH_{2j-1}(\chi)$, and the Hermite polynomials of different orders are mutual orthogonal, if and only if

$$D^* - \tau^2 D + C\tau = 0, \tag{64}$$

can equation (63) be true, which gives

$$\tau_\pm = \frac{C \pm \sqrt{C^2 + 4|D|^2}}{2D}, \tag{65}$$

and the energy is

$$E_\pm = 2jD\tau_\pm - jC = \pm j\sqrt{C^2 + 4|D|^2}. \tag{66}$$

Hence the Hamiltonian possesses $|\tau_+\rangle\rangle$ and $|\tau_-\rangle\rangle$ as its eigenvectors with the eigenvalues E_+ and E_-, respectively.

In summary,we have introduced the new wave functions associated with atomic coherent states in the EPR entangled state representation, which is proportional to the one-variable Hermite polynomials. We have also derived the Wigner functions of the atomic coherent states with angular momentum values j, which are proportional to the two-variable Hermite polynomials. This brings much convenience for studying the relationship between the atomic coherent states with different j, since the recursive properties of these two-kind of polynomials are well-known. In so doing, a new classical-quantum correspondence scheme for angular momentum system is established, and the quantum optics theory regarding to angular momentum system is enriched. We expect the result in this work can be further used in quantum optics and nuclear physics in the future works.

Appendix

To see that (47) is just the two-mode Wigner operator, we use the normally ordered form of vacuum state projection operator

$$|00\rangle\langle 00| =: \exp[-a_1^\dagger a_1 - a_2^\dagger a_2] :,$$

and the IWOP technique to performed the integration in (47) and obtain its explicit normally ordered form,

$$\begin{aligned}\Delta(\gamma,\rho) = \int \frac{\mathrm{d}^2\zeta}{\pi^3} &: \exp\{-\frac{|\gamma-\zeta|^2 + |\gamma+\zeta|^2}{2} + \rho\zeta^* - \zeta\rho^* \\ &+ (\gamma-\zeta)\, a_1^+ + (\gamma-\zeta)^*\, a_2^+ + (\gamma+\zeta)^*\, a_1 \\ &+ (\gamma+\zeta)\, a_2 - a_1^\dagger a_1 - a_2^\dagger a_2 - a_1^\dagger a_2^\dagger - a_1 a_2\} : \\ = \pi^{-2} &: \exp\{-|\rho|^2 - |\gamma|^2 + \gamma(a_1^\dagger + a_2) \\ &+ \gamma^*(a_2^\dagger + a_1) + \rho(a_1^\dagger - a_2) \\ &+ \rho^*(a_1 - a_2^\dagger) - 2a_1^\dagger a_1 - 2a_2^\dagger a_2\} :,\end{aligned} \tag{67}$$

let

$$\gamma = \alpha + \beta^*, \qquad \rho = \alpha - \beta^*,$$
$$\alpha = \frac{1}{\sqrt{2}}(x_1 + \mathrm{i}p_1), \quad \beta = \frac{1}{\sqrt{2}}(x_2 + \mathrm{i}p_2),$$

we see

$$\Delta(\gamma,\rho) = \Delta(\alpha,\alpha^*)\Delta(\beta,\beta^*),$$

which is just the product of two independent single-mode Wigner operators.

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